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ADDITIVELY AND MULTIPLICATIVELY INVERSE NEAR-SEMIRINGS

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ABSTRACT. It has been shown that in a near-semiring $(S, +, .)$ with $(S, +)$ as an inverse semigroup, the near-semiring S is strongly regular if and only if S is regular and reduced. In a near-semiring $(S, +, .)$ with $(S, +)$ as an inverse semigroup, equivalent conditions are obtained such that $(S, .)$ is also an inverse semigroup.

1. INTRODUCTION

A near-semiring is a nonempty set S with two binary operations '+' and '.' such that

(1) $(S, +)$ is a commutative semigroup with identity 0,
(2) $(S, .)$ is a semigroup,
(3) $(x + y)z = xz + yz$ for all $x, y, z \in S$.

The class of near-semirings contains the class of rings and abelian near-rings. Hence the class of near-semirings is the most generalized algebraic structure with two binary operations. Let $(\Gamma, +)$ be any commutative semigroup with identity 0. If $M(\Gamma)$ is the set of all mappings from Γ into Γ then $M(\Gamma)$ is a near-semiring under pointwise addition and composition. $M(\Gamma)$ is neither a ring, nor a near-ring, nor a semiring.

The semigroup $(S, +)$ is an inverse semigroup if for each $a \in S$, there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. Then a' is said to be additive inverse of a. A semiring $(R, +, .)$ is an additive inverse semiring if $(R, +)$ is an inverse semigroup. A near-semiring $(S, +, .)$ is an additive inverse near-semiring if $(S, +)$ is an inverse semigroup.

Bandelt and Petrich [2] have studied additive inverse semiring with the conditions $a(a + a') = a + a'$, $a(b + b') = (b + b')a$ and $a + a(b + b') = a$. Sen and

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Maity [9] have obtained equivalent conditions for an additive inverse semiring to be regular. In this paper we extend these results to near-semirings. We have obtained equivalent conditions for an additive inverse near-semiring $(S, +, .)$ such that the semigroup $(S, .)$ is also an inverse semigroup.

2. Strongly regular additively inverse near-semirings

Lemma 2.1. *For $x, y \in S$, $x = (x')'$, $(x+y)' = x' + y'$ and $(xy)' = x'y$.*

Proof: Straightforward.

If S is an additive inverse semiring then $(xy)' = x'y = xy'$ and $xy = x'y'$. We have $E^+(S) = \{a \in S | a + a = a\}$ and $E^\bullet(S) = \{e \in S | e.e = e\}$.

Bandelt, Petrich [2] and Sen, Maity [9] have studied additive inverse semiring that satisfies the following conditions:

(1) $a(a + a') = a + a'$
(2) $a(b + b') = (b + b')a$
(3) $a + a(b + b') = a$.

Throughout this paper we assume that additive inverse near-semiring satisfies $a(b+b') = (b+b')a$. We call such an additive inverse near-semiring as idempotent commuting additive inverse near-semiring. Rings and zero-symmetric near-rings are natural examples of these types of near-semirings. A nonempty subset I of S such that $a + b \in I$ for all $a, b \in I$ is said to be invariant subnear-semiring if $IS \subseteq I$ and $SI \subseteq I$.

Lemma 2.2. *$E^+(S)$ is an invariant subnear-semiring of an idempotent commuting additive inverse near-semiring S.*

Proof: Let $a, b \in E^+(S)$. Then clearly $a + b \in E^+(S)$. Let $s \in S$. Now $as + as = (a + a)s = as$. Therefore $as \in E^+(S)$. Since $a \in E^+(S)$ and inverse of any additive idempotent element is itself, we have $a = a'$. Now $sa = s(a + a) = s(a + a') = (a + a')s = as + a's = as + as \in E^+(S)$.

Sen and Maity [9] studied additively inverse semirings and derived equivalent conditions for an additive inverse semiring to be regular. Now we introduce strongly regular additive inverse near-semirings and characterize them.

Definition 2.1. *A near-semiring S is said to be reduced if for every $a \in S$, $a^n \in E^+(S)$ implies $a \in E^+(S)$ for any positive integer n.*

Definition 2.2. *A near-semiring S is said to be regular if for each $a \in S$ there exists an element $x \in S$ such that $a = axa$.*

Definition 2.3. *A near-semiring S is said to be strongly regular if for each $a \in S$ there exists an element $x \in S$ such that $a = xa^2$.*

Lemma 2.3. *Let S be a reduced idempotent commuting additive inverse near-semiring. Then for any $a, b \in S$, $ab \in E^+(S)$ implies $ba \in E^+(S)$and $asb \in E^+(S)$ for every $s \in S$.*

Proof: *Let $ab \in E^+(S)$. Now $(ba)^2 = baba \in E^+(S)$. Thus $ba \in E^+(S)$. Also $(asb)^2 = asbasb \in E^+(S)$, showing that $asb \in E^+(S)$.*

Lemma 2.4. *Let S be an additive inverse near-semiring and $a, b \in S$. If $a + b' \in E^+(S)$ and $a + a' = b + b'$, then $a = b$.*

Proof: Let $a, b \in S$. Now
$a+b' = (a+b')+(a+b')' = a+b'+b+a' = a+a'+b+b' = b+b'+b+b' = b+b'$.
Thus $a + b' + b = b + b' + b = b$. Hence $b = a + b' + b = a + a' + a = a$.

Lemma 2.5. *Let S be a reduced idempotent commuting additive inverse near-semiring. For any $a, b \in S$ and for any $e \in E^\bullet(S)$, $abe = aeb$.*

Proof: Let $e \in E^\bullet(S)$. Then for any $a, b \in S$,
$(a + (ae)')e = ae + (ae)'e = ae + (aee)' = ae + (ae)' \in E^+(S)$.
Since S is reduced, $abe + (aebe)' \in E^+(S)$. Now
$abe + (abe)' = (ab + (ab)')e = e(ab + a'b)e = e(a + a')be = aebe + (aebe)'$.
Therefore by Lemma 2.4, $abe = aebe$. Also
$(eb + (ebe)')e = ebe + (ebe)' \in E^+(S)$.
Hence $eb(eb + (ebe)') \in E^+(S)$ and $(ebe)'(eb + (ebe)') \in E^+(S)$.Thus
$(eb + (ebe)')^2 \in E^+(S)$. Since S is reduced, $eb + (ebe)' \in E^+(S)$.Now

$$\begin{aligned} eb + (eb)' &= eeb + (eeb)' = eeb + e'eb = (e + e')eb \\ &= e(e + e')b = e(eb + (eb)') = (eb + (eb)')e = ebe + (ebe)', \end{aligned}$$

showing that $ebe = eb$. Thus $abe = aeb$.

Note: If S is a reduced idempotent commuting additive inverse near-semiring with identity then the idempotents are central.

Lemma 2.5 does not hold for additive inverse near-semiring which does not satisfy the condition $a(b + b') = (b + b')a$, as the following example shows.

Example 2.1. Let $\Gamma = \{0, 1\}$ in which '+' is defined by

$+$	0	1
0	0	1
1	1	1

Now Γ is an additive inverse commutative semigroup. Let $M(\Gamma) = \{0, a, b, 1\}$ where $0, a, b, 1$ are all maps from Γ to Γ. Now $M(\Gamma)$ is an additive inverse near-semiring under pointwise addition and composition and we have

+	0	a	b	1
0	0	a	b	1
a	a	a	b	b
b	b	b	b	b
1	1	b	b	1

.	0	a	b	1
0	0	0	0	0
a	b	1	0	a
b	b	b	b	b
1	0	a	b	1

Clearly $M(\Gamma)$ is reduced. Here b is an idempotent with $aab \neq aba$, since $0 = a(b+b') \neq (b+b')a = b$.

Example 2.2. Let Γ be any additive inverse commutative semigroup with 0. (Let Γ be as in Example 2.1. Now $\Gamma \times Z$ is an infinite additive inverse commutative semigroup). Let $S_0(\Gamma) = \{f : \Gamma \to \Gamma | f(0) = 0\}$ and let
$S^{(1)}(\Gamma) = \{f_1 + f_2 + ... + f_k | f_i \in S_0(\Gamma)$ and $f_i(g + g^{'}) = (g + g^{'})f_i$ for all $g \in S_0(\Gamma)\}$. Then $S^{(1)}(\Gamma)$ is an additive inverse near-semiring under pointwise addition and composition. Clearly $S^{(1)}(\Gamma)$ is neither a ring, nor a near-ring, nor a semiring. But $S^{(1)}(\Gamma)$ is only an additive inverse near-semiring with $a(b+b') = (b+b')a$ for all $a, b \in S^{(1)}(\Gamma)$.

Theorem 2.1. *An idempotent commuting additive inverse near-semiring S is strongly regular if and only if it is regular and reduced.*

Proof: Let S be strongly regular and $a \in S$ be such that $a^2 \in E^+(S)$. Now there exists $x \in S$ such that $a = xa^2 \in E^+(S)$. Hence S is reduced.

Let us show that S is regular. Let $a \in S$. Then $a = xa^2$ for some $x \in S$. Hence $(a + (axa)')a = a^2 + a'xa^2 = a^2 + a'a = a^2 + (a^2)' \in E^+(S)$. Since S is reduced, $a(a + (axa)') \in E^+(S)$. Since $(a + axa')a \in E^+(S)$, $(a + axa')a'xa \in E^+(S)$. Hence $(a+axa')(axa)' \in E^+(S)$ and hence $(axa)'(a+(axa)') \in E^+(S)$. Now

$$\begin{aligned}(a + (axa)')^2 &= (a + (axa)')(a + (axa)') \\ &= a(a + (axa)') + (axa)'(a + (axa)') \in E^+(S).\end{aligned}$$

Hence $a + (axa)' \in E^+(S)$. Now

$$\begin{aligned}a + a' &= xa^2 + (xa^2)' = xa^2 + (xa)'a = (xa + (xa)')a = (xa + (xa)')xa^2 \\ &= x(xa + (xa)')a^2 = x(xa^2 + (xa^2)')a = (a + a')xa = axa + (axa)'.\end{aligned}$$

Hence $a = axa$ showing that S is regular.

Conversely let us assume that S is regular and reduced. Let $a \in S$. Then $a = aya$ for some $y \in S$. Clearly ya is an idempotent. Hence by Lemma 2.5, we have $a = aya = ayaya = ayyaa = ay^2a^2 = xa^2$, where $x = ay^2$. Thus S is strongly regular.

Theorem 2.2. *An idempotent commuting additive inverse near-semiring S is strongly regular if and only if given $a \in S$ there exists $x \in S$ such that $a = axa$ and $ax = xa$.*

Proof: Assume that S is strongly regular. Let $a \in S$, such that $a = xa^2$, for some $x \in S$. By Theorem 2.1, $a = axa$. Now $(ax + (xa)')a = a + (xa^2)' = a + a' \in E^+(S)$. Since S is reduced, $ax(ax+(xa)') \in E^+(S)$. Since $(ax+(xa)')a \in E^+(S)$, $(ax + (xa)')x'a \in E^+(S)$. Therefore $(ax + (xa)')(xa)' \in E^+(S)$ and hence $(xa)'(ax + (xa)') \in E^+(S)$. Thus $(ax + (xa)')^2 \in E^+(S)$ and hence $(ax + (xa)') \in E^+(S)$. Also

$$
\begin{aligned}
ax + (ax)' &= (a + a')x = (xa^2 + (xa^2)')x = (xa + (xa)^{'})ax \\
&= a(xa + (xa)')x = a(xax + (xax)^{'}) = (xax + (xax)^{'})a = xa + (xa)'.
\end{aligned}
$$

Therefore $ax = xa$. The converse is immediate.

Corollary 2.1. *(*[8]*, Theorem* 9.158*) Let $N \neq \{0\}$ be a regular near-ring with identity. The following statements are equivalent.*

(1) $N = N_0$ has no non-zero nilpotent elements.
(2) All idempotents of N are central.

Proof: If N is a near-ring, then $E^+(N) = 0$.

3. Multiplicatively inverse near-semirings

Definition 3.1. *An element $a \in S$ is a weak idempotent if $a^2 = a + y$ for some $y \in E^+(S)$. The set of weak idempotents of S is denoted by $E^*(S)$.*

If $a \in (S, +, .)$ is an idempotent then a is a weak idempotent.

Now we give an example of a weak idempotent element which is not an idempotent.

Following Alarcon and Polkowska [1], we have the following definition for $B(n, i)$ semirings without zero. Let $n \geq 2$ and $1 \leq i \leq n$ and $m = n - i$. Let $B(n, i)$ be the following semirings. $B(n, i) = \{1, 2, ..., n-1\}$ and the operations in $B(n, i)$ are:

$$
x +_{B(n,i)} y = \begin{cases} x + y \text{ if } x + y \leq n - 1 \\ l \text{ if } x + y \geq n \\ \text{with } l = x + y \bmod m \text{ and } i \leq l \leq n - 1. \end{cases}
$$

$$
x \cdot_{B(n,i)} y = \begin{cases} xy \text{ if } xy \leq n - 1 \\ l \text{ if } xy \geq n \\ \text{with } l = xy \bmod m \text{ and } i \leq l \leq n - 1. \end{cases}
$$

Example 3.1. Consider the semiring $B(4,3) = \{1,2,3\}$ where '+' and '.' are defined as follows:

+	1	2	3
1	2	3	3
2	3	3	3
3	3	3	3

.	1	2	3
1	1	2	3
2	2	3	3
3	3	3	3

Here 2 is a weak idempotent but not an idempotent, since $2.2 = 2+3$ for $3 \in E^+(B(4,3))$ but $2.2 = 3 \neq 2$.

Definition 3.2. A near-semiring $(S,+,.)$ is a multiplicative inverse near-semiring if $(S,.)$ is an inverse semigroup.

Definition 3.3. Let $(S,+,.)$ be a near-semiring. Let $a \in S$. If there exists a unique $x \in S$ such that $axa = a+y_1$ and $xax = x+y_2$ for some $y_1, y_2 \in E^+(S)$ then x is called a multiplicative weak inverse of a.

If every element $a \in S$ has multiplicative weak inverse then $(S,+,.)$ is called multiplicative weak inverse near-semiring. Multiplicative weak inverse $a^{'}$ of a weak idempotent a is a itself.

Remark 3.1. If $(S,+,.)$ is a multiplicative inverse near-semiring then it is a multiplicative weak inverse near-semiring. But the converse is not true.

Now we give an example of an element which has multiplicative weak inverse but does not have a multiplicative inverse.

Example 3.2. Consider the near-semiring $(S,+,.)$ where '+' and '.' are defined as follows:

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	b	a	b	d	d
c	c	a	d	c	d
d	d	a	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	d
c	0	a	b	b	d
d	0	a	b	b	d

Here c has a multiplicative weak inverse a, since $cac = c+a$ and $aca = a+a$ for $a \in E^+(S)$. But c does not have a multiplicative inverse.
Hereafter we assume that for any $a, b \in S$, $y \in E^+(S)$, $a(b+y) = ab + ay$. Clearly zero-symmetric near-rings, semirings and rings satisfy this condition.

Lemma 3.1. *If S is a multiplicative weak inverse near-semiring then for any $e, f \in E^*(S)$, $ef = fe$.*

Proof: Let $e^2 = e + y_1$ and $f^2 = f + y_2$ for some $y_1, y_2 \in E^+(S)$. Let x be the multiplicative weak inverse of ef. Then $ef(x)ef = ef + y_3$ and $x(ef)x = x + y_4$ for some $y_3, y_4 \in E^+(S)$.
$(fxe)^2 = f(xefx)e = f(x + y_4)e = f(xe + y_4e) = fxe + fy_4e = fxe + y_5$
for some $y_5 \in E^+(S)$. Hence fxe is a weak idempotent.

We have
$ef(fxe)ef = ef^2xe^2f = e(f + y_2)x(e + y_1)f = ef + y_6$ for some $y_6 \in E^+(S)$.
$fxe(ef)fxe = fxe^2f^2xe = fx(e + y_1)(f + y_2)xe = fxe + y_7$ for some $y_7 \in E^+(S)$. Thus $fxe = ef$. Therefore ef is a weak idempotent.
Similarly fe is also a weak idempotent.
Now $ef(fe)ef = ef^2e^2f = e(f + y_2)(e + y_1)f = efef + y_8 = ef + y_9$ for some $y_8, y_9 \in E^+(S)$.
$fe(ef)fe = fe^2f^2e = f(e + y_1)(f + y_2)e = fe(f + y_2)e + y_{10} = fefe + y_{11} = fe + y_{12}$ for some $y_{10}, y_{11}, y_{12} \in E^+(S)$.
Thus fe is the multiplicative weak inverse of ef.

Since ef is the multiplicative weak inverse of the weak idempotent ef, we have $ef = fe$.

Definition 3.4. *The invariant subnear-semiring $E^+(S)$ is k-invariant if $a + y$ and $y \in E^+(S)$ imply $a \in E^+(S)$.*

Theorem 3.1. *Let S be a multiplicative weak inverse near-semiring such that $E^+(S)$ is k-invariant. For any $a \in S$, $a^2 \in E^+(S)$ implies $a \in E^+(S)$.*

Proof: Let b be the multiplicative weak inverse of a. Thus $aba = a+y_1$ and $bab = b + y_2$ for some $y_1, y_2 \in E^+(S)$. Thus ab and ba are weak idempotents. Hence by Lemma 3.1, $ab^2a = abba = baab \in E^+(S)$.

Now
$$a(ba(ba + b))a = aba(ba + b)a = (a + y_1)(ba^2 + ba)$$
$$= a(ba^2 + ba) + y_3 = aba + y_4 = a + y_5$$
for some $y_3, y_4, y_5 \in E^+(S)$.

Now
$$\begin{aligned}(ba(ba + b))a(ba(ba + b)) &= ba(ba + b)(a + y_1)(ba + b) = ba(ba + b)a(ba + b) + y_6\\ &= ba(ba^2 + ba)(ba + b) + y_6 = (ba + y_7)(ba + b) + y_6\\ &= ba(ba + b) + y_8\end{aligned}$$
for some $y_6, y_7, y_8 \in E^+(S)$.

By uniqueness, $ba(ba + b) = b$

Now $babba = bbaab = b^2a^2b \in E^+(S)$. Then
$babba = (b + y_2)ba = b^2a + y_2ba \in E^+(S)$. Since $y_2ba \in E^+(S)$, $b^2a \in E^+(S)$.

Now

$$ab^2a = abba = aba(ba + b)ba = (a + y_1)(ba + y_9 + b^2a)$$
$$= aba + ab^2a + y_{10} = a + y_1 + ab^2a + y_{10} = a + y_{11}$$

for some $y_9, y_{10}, y_{11} \in E^+(S)$. Since $ab^2a \in E^+(S)$, $a \in E^+(S)$.

Theorem 3.2. *Let $(S, +, .)$ be an idempotent commuting additive inverse near-semiring with $E^+(S)$ as k-invariant. Then the following are equivalent:*

(1) *$(S, .)$ is an inverse semigroup.*
(2) *$(S, .)$ is regular and idempotents in $E^\bullet(S)$ are central.*
(3) *$(S, .)$ is regular and $aS = Sa$ for every $a \in S$.*

Proof: (1) $\Rightarrow$ (2) Clearly $(S, .)$ is regular. Let $a \in S$ and $e \in E^\bullet(S)$ and $a = aba$ for some $b \in S$. Let $ab = f$. Then $a = fa$. By Theorem 3.1, S is reduced. By Lemma 2.5, $ae = fae = fea = efa = ea$.
(2) $\Rightarrow$ (3) Let $a \in S$ and let $a = axa$ for some $x \in S$. For any $s \in S$, $as = axas = asxa \in Sa$. Thus $aS \subseteq Sa$. Similarly $Sa \subseteq aS$. Thus $aS = Sa$.
(3) $\Rightarrow$ (1) Let $e, f \in E^\bullet(S)$. Now $eS = Se$. Hence there exists $x, y \in S$ such that $fe = ex$ and $ef = ye$. Hence $efe = eex = ex = fe$ and $efe = yee = ye = ef$. Therefore $ef = fe$. By Theorem 1.17 [3],$(S, .)$ is an inverse semigroup.

Corollary 3.1. *(*[7], Theorem1*)If $(N, +, .)$ is a near-ring then the following are equivalent:*

(1) *$(N, .)$ is an inverse semigroup.*
(2) *$(N, .)$ is regular and idempotents are central.*
(3) *$(N, .)$ is regular and $Na = aN$ for every $a \in N$.*

Proof: If $(N, +, .)$ is a near-ring then clearly $E^+(N) = \{0\}$ is k-invariant and $a(b + y) = ab + ay$ for all $a, b \in N$ and $y \in E^+(N)$.

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